abc=4RF without trigonometry.

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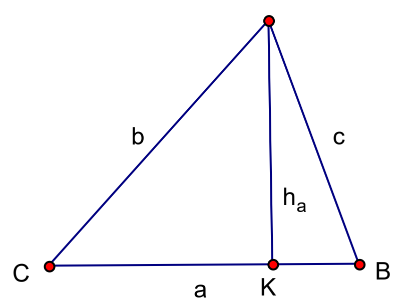
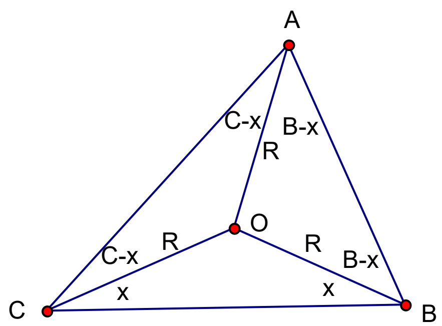
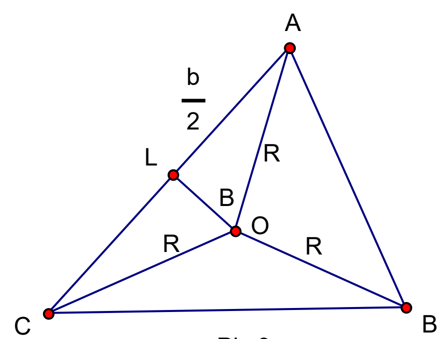


Pic.1

Frst note that $F := [ABC] = \frac{ah_a}{2}$ and similarly $F = \frac{bh_b}{2} = \frac{ch_c}{2}$.



Pic.2

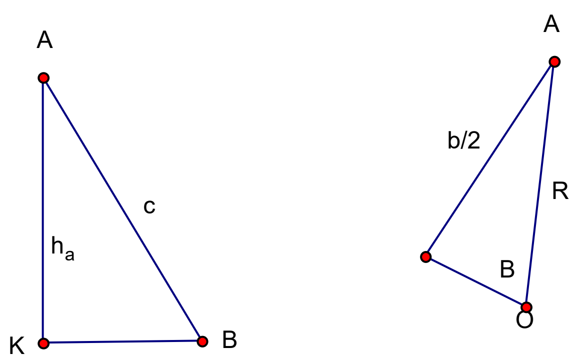


Pic.3

Let $x := \angle OCB$. (Pic.2). Then $\angle OBC = x$ and as well as $\angle OCA = \angle OAC = C - x$, $\angle OBA = \angle OAB = B - x$.

Hence $A = C - x + B - x \Leftrightarrow 2x = B + C - A$ and, therefore,

$\angle BOC = 180^\circ - 2x = 180^\circ - B - C + A = 2A$. Similarly, $\angle COA = 2B, \angle AOB = 2C$.



Pic.4

Since $\angle AOL = B$ (Pic.3) then right triangles $\triangle AKB$ and $\triangle ALO$ (Pic.4) are similar.
Then $\frac{h_a}{c} = \frac{b/2}{R} \Leftrightarrow h_a = \frac{bc}{2R} \Leftrightarrow bc = 2h_a R \Rightarrow abc = 2ah_a R = 4RF$.
So, we obtain without trig. two important correlations:

1. $h_a = \frac{bc}{2R}$ (which express length height dropped on a via circumradius and two other sidelengths;

2. $abc = 4RF$ that give opportunity find value of circumradius via sidelengths of the triangle.

Remark 1.
In the case of introduction to trig. of acute angles we also immediately obtain complete SineTheorem.
Indeed, then $\frac{b/2}{R} = \sin B \Leftrightarrow 2R = \frac{b}{\sin B}$ and similar $2R = \frac{a}{\sin A} = \frac{c}{\sin C}$.
Since $abc = 4RF$ then $2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{abc}{2F}$.

Problem
★ Known that in a triangle $F = \frac{bc}{2}$. Prove, without using trig., that triangle is right angled.

Solution.
Since $F = \frac{ah_a}{2}$ and $h_a = \frac{bc}{2R}$ then $ah_a = bc \Rightarrow a = 2R \Rightarrow \angle A = 90^\circ$.
(Another proof:
$F = \frac{bc}{2} \Leftrightarrow 2bc = 4F \Leftrightarrow 4b^2c^2 = 16F^2 \Leftrightarrow 4b^2c^2 = 4b^2c^2 - (b^2 + c^2 - a^2)^2 \Leftrightarrow b^2 + c^2 = a^2$)

Remark 2. Traditional way to derive $abc = 4RF$.
Since $F = \frac{bc\sin A}{2}$ and $a = 2R\sin A$ then $4FR = bc \cdot 2R\sin A = abc$